Signals & Systems Cheat Sheet[1]

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Introduction

C-T/D-T Signal Models, System Properties

Unit step function:

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Unit ramp function:

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Relationship between $u(t)$ and $r(t)$:

$r(t) = \int_{-\infty}^{t} u(\lambda)\,\mathrm{d}\lambda$
$u(t) = \frac{\mathrm{d}}{\mathrm{d}t} r(t)$

Dirac Delta function:

$\int_{-\varepsilon}^{\varepsilon} \delta(t)\,\mathrm{d}t = 1 \quad \forall \varepsilon > 0$

Sifting property of the delta function:

$$\int_{t_0-\varepsilon}^{t_0+\varepsilon} f(t) \cdot \delta(t - t_0)\,\mathrm{d}t = f(t_0) \quad \forall \varepsilon > 0$$

Relationship between $\delta(t)$ and $u(t)$:

$\int_{-\infty}^{t} \delta(\lambda)\,\mathrm{d}\lambda = u(t), \quad t \neq 0$
$\delta(t) = \frac{\mathrm{d}}{\mathrm{d}t} u(t)$

Rectangular pulse function (width τ):

$$p_\tau(t) = \begin{cases} 1, & -\tau/2 \leq t \leq \tau/2 \\ 0, & \text{otherwise} \end{cases}$$

$p_\tau(t) = u(t + \tau/2) - u(t - \tau/2)$

Discrete-time (D-T) signal:

Sequence of numbers indexed by integers
$x[n] \quad n = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

Sampling a continuous-time signal:

$y[n] = y(t)|_{t=n \cdot T} = y(n \cdot T)$
T: sampling interval
$F_\mathrm{s} \equiv 1/T$: sampling rate

Discrete-time sinusoid:

$x(t) = A \cdot \cos(2\pi f_0 \cdot t)$
$\Rightarrow x[n] = x(t = n \cdot T) = A \cdot \cos(2\pi f_0 \cdot nT)$

$$x[n] = A \cdot \cos(\Omega_0 \cdot n),\ \Omega_0 = 2\pi \cdot \frac{f_0}{F_\mathrm{s}},\ [\Omega_0] = \mathrm{rad}/\mathrm{sample}$$

General D-T sinusoid:

$x[n] = A \cdot \cos(\Omega_0 \cdot n + \theta),\ \Omega_0 \equiv \Omega_0 \pm 2\pi, \pm 4\pi, \ldots$

Discrete-time impulse or D-T delta:

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Sifting property of the D-T delta function:

$$\sum_{n=-\infty}^{\infty} \delta[n] = 1,\ \sum_{n=-\infty}^{\infty} x[n] \cdot \delta[n - n_0] = x[n_0]$$

D-T rectangular pulse ($2q + 1$ samples):

$$p_q[n] = \begin{cases} 1, & n = -q, \ldots, -1, 0, 1, \ldots, q \\ 0, & \text{otherwise} \end{cases}$$

Causality:

A causal system's output at a time t_1 does not depend on values of the input $x(t)$ for $t > t_1$.

Linearity ($\rightarrow$ Superposition):

$x_1(t) \rightarrow y_1(t) \qquad x_2(t) \rightarrow y_2(t)$
$\Rightarrow x(t) = a_1 \cdot x_1(t) + a_2 \cdot x_2(t) \rightarrow y(t) = a_1 \cdot y_1(t) + a_2 \cdot y_2(t)$

Time invariance:

$x(t) \rightarrow y(t) \Rightarrow x(t - t_0) \rightarrow y(t - t_0)$

LTI: Linear Time-Invariant systems

Diff. Equations

C-T System Model: Differential Equations,
D-T Signal Model: Difference Equations

Linear constant-coefficient differential equations of order N:

$y^{(N)}(t) + \sum_{i=0}^{N-1} a_i \cdot y^{(i)}(t) = \sum_{i=0}^{M} b_i \cdot x^{(i)}(t)$
Solution: "zero-input response" (due to I.C.s) + "zero-state response" (due to input): $y(t) = y_{\mathrm{ZI}}(t) + y_{\mathrm{ZS}}(t)$

Linear constant coefficient difference equation of order N:

$y[n] + a_1 \cdot y[n-1] + \cdots + a_N \cdot y[n-N] =$
$b_0 \cdot x[n] + b_1 \cdot x[n-1] + \cdots + b_M \cdot x[n-M]$

Solution of a difference equation by recursion:

$y[n] = -a_1 \cdot y[n-1] - \cdots - a_N \cdot y[n-N] + b_0 \cdot x[n] +$
$b_1 \cdot x[n-1] + \cdots + b_M \cdot x[n-M]$

Convolution

C-T System Model: Convolution Integral,
D-T System Model: Convolution Sum

D-T convolution:

$$y[n] = x[n] * h[n] = \sum_{i=-\infty}^{\infty} x[i] \cdot h[n-i]$$

$h[n]$: impulse response of an LTI D-T system
$y[n]$: zero state solution
$x[n]$: input

Properties of convolution (both D-T and C-T):

Commutativity:
$x[n] * h[n] = h[n] * x[n]$
Associativity:
$x[n] * (v[n] * w[n]) = (x[n] * v[n]) * w[n]$
Distributivity:
$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$

C-T convolution:

$$y(t) = x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t - \tau)\,\mathrm{d}\tau$$

C-T conv. derivative property: $y(t) = x(t) * h(t)$

$\frac{\mathrm{d}}{\mathrm{d}t} y(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t} x(t)\right) * h(t) = x(t) * \left(\frac{\mathrm{d}}{\mathrm{d}t} h(t)\right)$

C-T conv. integration property: $y(t) = x(t) * h(t)$

$\int_{-\infty}^{t} y(\lambda)\,\mathrm{d}\lambda = \left[\int_{-\infty}^{t} x(\lambda)\,\mathrm{d}\lambda\right] * h(t) =$
$= x(t) * \left[\int_{-\infty}^{t} h(\lambda)\,\mathrm{d}\lambda\right]$

[1] https://github.com/m-thu/sandbox/blob/master/signals-systems.tex

CT Fourier Signal Models

Fourier Series, Fourier-Transform (CTFT)

Fourier series (Complex exp. form):

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \cdot \mathrm{e}^{\mathrm{j}\,k\omega_0 \cdot t}$$

Fourier series (Trig. form):

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cdot \cos(k\omega_0 \cdot t + \theta_k)$$

Complex Fourier series coefficients:

$$c_k = \frac{1}{T} \cdot \int_{t_0}^{t_0+T} x(t) \cdot \mathrm{e}^{-\mathrm{j}\,k\omega_0 \cdot t}\,\mathrm{d}t$$

Trigonometric FS with complex coefficients:

$x(t) = c_0 + \sum_{k=1}^{\infty} 2 \cdot |c_k| \cdot \cos(k\omega_0 \cdot t + \angle c_k)$

Fourier transform:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot \mathrm{e}^{-\mathrm{j}\,\omega t}\,\mathrm{d}t$$

Inverse Fourier transform:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot \mathrm{e}^{\mathrm{j}\,\omega t}\,\mathrm{d}\omega$$

Definition of the sinc-function:

$$\mathrm{sinc}(x) = \frac{\sin(\pi \cdot x)}{\pi \cdot x}$$

Timelimited signal:

$x(t) = 0 \quad \forall t \notin [T_1, T_2]$

Bandlimited signal:

$|X(\omega)| = 0 \quad \forall |\omega| > 2\pi \cdot B$

Linearity of the FT:

$a \cdot x(t) + b \cdot y(t) ○\!\!-\!\!\bullet\; a \cdot X(\omega) + b \cdot Y(\omega)$

Time shift:

$x(t-c) ○\!\!-\!\!\bullet\; X(\omega) \cdot \mathrm{e}^{-\mathrm{j}\omega c}$

Complex exponential modulation property:

$x(t) \cdot \mathrm{e}^{\mathrm{j}\omega_0 \cdot t} ○\!\!-\!\!\bullet\; X(\omega - \omega_0)$

Real sinusoid modulation:

$x(t) \cdot \cos(\omega_0 \cdot t) ○\!\!-\!\!\bullet\; \frac{1}{2} \cdot [X(\omega + \omega_0) + X(\omega - \omega_0)]$
$x(t) \cdot \sin(\omega_0 \cdot t) ○\!\!-\!\!\bullet\; \frac{\mathrm{j}}{2} \cdot [X(\omega + \omega_0) - X(\omega - \omega_0)]$

Convolution property:

$$x(t) * h(t) ○\!\!-\!\!\bullet\; X(\omega) \cdot H(\omega)$$

Differentiation property:

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n} \cdot x(t) ○\!\!-\!\!\bullet\; (\mathrm{j}\omega)^n \cdot X(\omega)$$

Fourier transform of the delta function:

$\delta(t) ○\!\!-\!\!\bullet\; 1$
$1 ○\!\!-\!\!\bullet\; 2\pi \cdot \delta(\omega)$

Fourier transform of sine and cosine:

$\cos(\omega_0 \cdot t) ○\!\!-\!\!\bullet\; \pi \cdot [\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$
$\sin(\omega_0 \cdot t) ○\!\!-\!\!\bullet\; \mathrm{j}\pi \cdot [\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$

FT of a periodic signal (c_k: Fourier series coefficients):

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi \cdot c_k \cdot \delta(\omega - k \cdot \omega_0)$$

CT Fourier System models

Frequency response

Response of an LTI system to a sinusoid:

$$x(t) = A \cdot \cos(\omega_0 \cdot t + \theta)$$

$$\to y(t) = |H(\omega_0)| \cdot A \cdot \cos\left(\omega_0 \cdot t + \theta + \angle H(\omega_0)\right)$$

Response to periodic inputs:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k^{(x)} \cdot \mathrm{e}^{\mathrm{j}\,k \cdot \omega_0 t}$$

$$\to y(t) = \sum_{k=-\infty}^{\infty} H(k \cdot \omega_0) \cdot c_k^{(x)} \cdot \mathrm{e}^{\mathrm{j}\,k \cdot \omega_0 t}$$

Output Fourier series coefficients:

$c_k^{(y)} = H(k \cdot \omega_0) \cdot c_k^{(x)}$

Response to aperiodic signals:

$$x(t) = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} X(\omega) \cdot \mathrm{e}^{\mathrm{j}\,\omega t}\,\mathrm{d}\omega$$

$$\to y(t) = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} H(\omega) \cdot X(\omega) \cdot \mathrm{e}^{\mathrm{j}\,\omega t}\,\mathrm{d}\omega$$

Ideal low-pass filter (t_d: filter delay in time domain):

$$H(\omega) = \begin{cases} 1 \cdot \mathrm{e}^{-\mathrm{j}\omega t_\mathrm{d}}, & -\Omega < \omega < \Omega \quad \text{(Passband)} \\ 0, & \text{otherwise} \quad \text{(Stopband)} \end{cases}$$

Ideal high-pass filter:

$$H(\omega) = \begin{cases} 0, & -\Omega < \omega < \Omega \quad \text{(Stopband)} \\ 1 \cdot \mathrm{e}^{-\mathrm{j}\omega t_\mathrm{d}}, & \text{otherwise} \quad \text{(Passband)} \end{cases}$$

Summary of *ideal* filters:

1. Magnitude response:
   a) Constant in passband.
   b) Zero in stopband.
2. Phase response:
   a) Linear in passband (neg. slope = delay).
   b) Undefined in stopband.

Laplace Models for CT Signals & Systems

Transfer Function

One-sided Laplace Transform:

$$X(s) = \int_{0^-}^{\infty} x(t) \cdot \mathrm{e}^{-st}\,\mathrm{d}t, \quad s = \sigma + \mathrm{j}\omega$$

Inverse Laplace Transform:

$x(t) = \frac{1}{2\pi\mathrm{j}} \cdot \int_{c-\mathrm{j}\infty}^{c+\mathrm{j}\infty} X(s) \cdot \mathrm{e}^{st}\,\mathrm{d}s, \quad s = c + \mathrm{j}\omega$ is in ROC

Linearity of the LT:

$a \cdot x(t) + b \cdot y(t) ○\!\!-\!\!\bullet\; a \cdot X(s) + b \cdot Y(s)$

Time shift ($c > 0$, $x(t)$ causal):

$x(t-c) ○\!\!-\!\!\bullet\; \mathrm{e}^{-cs} \cdot X(s)$

Time differentiation:

$$\dot{x}(t) ○\!\!-\!\!\bullet\; s \cdot X(s) - x(t = 0^-)$$

Integration:

$$\int_{0^-}^{t} x(\lambda)\,\mathrm{d}\lambda \circ\!\!-\!\!\bullet \frac{1}{s}\cdot X(s)$$

Convolution:

$$x(t) * h(t) \circ\!\!-\!\!\bullet X(s)\cdot H(s)$$

First-order linear ODE ($s+a$: characteristic polynomial):

$$\dot{y}(t) + a\cdot y(t) = b\cdot x(t)$$
$$\rightarrow Y(s) = \underbrace{\frac{y(0^-)}{s+a}}_{\text{zero input sol.}} + \underbrace{\frac{b}{s+a}\cdot X(s)}_{\text{zero state sol.}}$$

Second-order linear ODE ($s^2 + a_1\cdot s + a_0$: char. pol.):

$$\ddot{y}(t) + a_1\cdot\dot{y}(t) + a_0\cdot y(t) = b_1\cdot\dot{x}(t) + b_0\cdot x(t)$$

Causal input: $x(t) = 0, t < 0 \rightarrow x(0^-) = 0$

$$\rightarrow Y(s) = \underbrace{\frac{y(0^-)\cdot s + \dot{y}(0^-) + a_1\cdot y(0^-)}{s^2 + a_1\cdot s + a_0}}_{\text{zero-input sol.}} + \underbrace{\frac{b_1\cdot s + b_0}{s^2 + a_1\cdot s + a_0}\cdot X(s)}_{\text{zero-state sol.}}$$

Zero-input solution (2nd-order system): $|\zeta| < 1, \omega_\mathrm{n} > 0$

$$Y(s) = \frac{\alpha}{s^2 + 2\zeta\omega_\mathrm{n}\cdot s + \omega_\mathrm{n}^2}$$
$$\rightarrow y(t) = A\cdot \mathrm{e}^{-\zeta\omega_\mathrm{n}\cdot t}\cdot\sin\left(\left(\omega_\mathrm{n}\cdot\sqrt{1-\zeta^2}\right)\cdot t\right)\cdot u(t),$$
$$A = \frac{\alpha}{\omega_\mathrm{n}\cdot\sqrt{1-\zeta^2}}$$
$$Y(s) = \beta\cdot\frac{s+\alpha}{s^2 + 2\zeta\omega_\mathrm{n}\cdot s + \omega_\mathrm{n}^2}$$
$$\rightarrow y(t) = A\cdot \mathrm{e}^{-\zeta\omega_\mathrm{n}\cdot t}\cdot\sin\left(\left(\omega_\mathrm{n}\cdot\sqrt{1-\zeta^2}\right)\cdot t + \phi\right)\cdot u(t),$$
$$A = \beta\cdot\sqrt{\frac{(\alpha-\zeta\omega_\mathrm{n})^2}{\omega_\mathrm{n}^2\cdot(1-\zeta^2)} + 1},$$
$$\phi = \arctan\left(\frac{\omega_\mathrm{n}\cdot\sqrt{1-\zeta^2}}{\alpha-\zeta\omega_\mathrm{n}}\right)$$

N^{th}-order linear ODE ($M \leq N$):

$$y^{(N)}(t) + a_{N-1}\cdot y^{(N-1)}(t) + \cdots + a_1\cdot\dot{y}(t) + a_0\cdot y(t)$$
$$= b_M\cdot x^{(M)} + \cdots + b_1\cdot\dot{x}(t) + b_0\cdot x(t)$$

$$\rightarrow Y(s) = \frac{\mathrm{IC}(s)}{A(s)} + \underbrace{\frac{B(s)}{A(s)}}_{H(s)}\cdot\underbrace{X(s)}_{\frac{N_X(s)}{D_X(s)}}$$

$A(s) = s^N + a_{N-1}\cdot s^{N-1} + \cdots + a_1\cdot s + a_0,$
$B(s) = b_M\cdot s^M + \cdots + b_1\cdot s + b_0,$
$\mathrm{IC}(s) =$ polynomial in s dependent of ICs

Solution for general system with ICs (no common roots between denominator of $X(s)$ and $H(s)$):

$$Y(s) = \overbrace{\frac{\mathrm{IC}(s)}{A(s)}}^{\text{zero-input response}} + \overbrace{\underbrace{\frac{E(s)}{A(s)}}_{\text{transient response}} + \underbrace{\frac{F(s)}{D_X(s)}}_{\text{steady-state response}}}^{\text{zero-state response}}$$

Transfer function (system in zero-state):

$$Y(s) = \frac{B(s)}{A(s)}\cdot X(s) = H(s)\cdot X(s)$$

Connection between frequency response and transfer function (if $H(\mathrm{j}\omega)$ exists):

$$H(\mathrm{j}\omega) = H(s)|_{s=\mathrm{j}\omega}$$

$H(\mathrm{j}\omega)$ always exists for R, L, C circuits.

S-domain impedances:

$$Z_R(s) = R \qquad Z_C(s) = \frac{1}{sC} \qquad Z_L(s) = sL$$

Bounded-input, bounded-output (BIBO) stability:

For any bounded input $|x(t)| \leq C_1\ \forall t \geq 0$ the output remains bounded: $|y(t)| \leq C_2$.

Checks for BIBO stability:

- $\int_0^\infty |h(t)|\,\mathrm{d}t < \infty$
- All poles of $H(s)$ (N^{th}-order system with N poles $p_i = 1, 2, \ldots, N$) are in the open left-half of the s-plane: $\Re\{p_i\} < 0$ for all $i = 1, 2, \ldots, N$.

Marginally stable systems:

Systems that have at least one *distinct* pole on the jω axis, but no repeated poles.

Real poles p_i:

$h_i(t) = c_i\cdot\mathrm{e}^{p_i\cdot t}\cdot u(t)$ (distinct pole)
$h_i(t) = c_i\cdot t^{k-1}\cdot\mathrm{e}^{p_i\cdot t}\cdot u(t)$ (k-repeated poles)

Complex pole pairs $p_i = \sigma_i \pm \mathrm{j}\omega_i,\ \omega_i > 0$:

$h_i(t) = c_i\cdot\mathrm{e}^{\sigma_i\cdot t}\cdot\cos(\omega_i\cdot t + \theta_i)\cdot u(t)$ (distinct pole)
$h_i(t) = c_i\cdot t^{k-1}\cdot\mathrm{e}^{\sigma_i\cdot t}\cdot\cos(\omega_i\cdot t + \theta_i)\cdot u(t)$ (k-repeated poles)

Response of an (unstable) system with transfer function $H(s)$ to a sinusoid, no I.C.s:

$$x(t) = A\cdot\cos(\omega_0\cdot t)\cdot u(t)$$
$$\rightarrow y(t) = y_t(t) + \underbrace{A\cdot|H(\mathrm{j}\omega_0)|\cdot\cos\left(\omega_0\cdot t + \angle H(\mathrm{j}\omega_0)\right), t \geq 0}_{\text{steady-state response}}$$

$y_t(t)$: transient response