Digital Signal Processing Cheat Sheet[1]

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Introduction

CT/DT Signal Models

Unit step function:

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Sifting property of the delta function ($\varepsilon > 0$):

$$\int_{t_0-\varepsilon}^{t_0+\varepsilon} f(t)\,\delta(t-t_0)\,\mathrm{d}t = f(t_0)$$

Rectangular pulse function (τ: pulse width):

$$p_\tau(t) = \begin{cases} 1, & -\tau/2 \leq t \leq \tau/2 \\ 0, & \text{otherwise} \end{cases}$$

Sampling ($y(t)$: CT signal, $y[n]$: DT signal, T_s: Sampling time, $F_\mathrm{s} = 1/T_\mathrm{s}$: sampling rate):

$$y[n] = y(t)|_{t=nT_\mathrm{s}} = y(n \cdot T_\mathrm{s})$$

Unit pulse (DT impulse, DT delta):

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

DT rectangular pulse (q: $\pm$ pulse extent):

$$p_q[n] = \begin{cases} 1, & n = -q, \ldots, -1, 0, 1, \ldots, q \\ 0, & \text{otherwise} \end{cases}$$

DT sinusoid (Ω: DT frequency, $[\Omega] = \mathrm{rad}/\mathrm{sample}$):

$$x[n] = A \cdot \cos(\Omega \cdot n + \theta)$$

Difference Equations

DT System Models

General N^{th} order difference equation ($y[n]$: output, $x[n]$: input):

$$y[n] + a_1 y[n-1] + \cdots + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M]$$

Convolution

DT System Models

Solutions of DT LTI systems (difference equation):

"Zero input" solution $\rightarrow$ Char. roots of polynomial
$+$ "Zero state" solution $\rightarrow$ Convolution

Convolution ($x[n]$: input, $y[n]$: output, $h[n]$: impulse response):

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$

Length of output of linear convolution (L: length of first signal, M: length of second signal):

$$L + M - 1$$

Commutativity of convolution:

$$y[n] = x[n] * h[n] = h[n] * x[n]$$

DT Fourier Signal Models

DTFT, DFT&FFT

Sampling a CT signal ($x(t)$: CT signal, $X(\omega) = \mathcal{F}\{x(t)\}$):

$$\widetilde{X}(\omega) = \frac{1}{T_\mathrm{s}}\,[\cdots + X(\omega - 4\pi F_\mathrm{s}) + X(\omega - 2\pi F_\mathrm{s}) + X(\omega) + X(\omega + 2\pi F_\mathrm{s}) + X(\omega + 4\pi F_\mathrm{s}) + \ldots]$$

Nyquist frequency (B: highest input signal frequency):

$$F_\mathrm{s} \geq 2B$$

Definition of the DT frequency ($[\Omega] = \mathrm{rad}$):

$$\Omega \equiv \omega \cdot T_\mathrm{s} = \omega \cdot \frac{1}{T_\mathrm{s}}$$

Definition of the DTFT ($-\pi \leq \Omega < \pi$):

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] \cdot \mathrm{e}^{-\mathrm{j}n\Omega}$$

Definition of the inverse DTFT:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) \cdot \mathrm{e}^{\mathrm{j}n\Omega}\,\mathrm{d}\Omega$$

DTFT: timeshift ($q \in \mathbb{Z}$):

$$x[n-q] \circ\!\!-\!\!\bullet\, \mathrm{e}^{-\mathrm{j}q\Omega} \cdot X(\Omega)$$

Definition of the DFT ($x[n]$: time domain signal, $n = 0, \ldots, N-1$):

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot \mathrm{e}^{-\mathrm{j}\frac{2\pi kn}{N}} \qquad k = 0, 1, 2, \ldots, N-1$$

Definition of the inverse DFT ($X[k]$: frequency domain signal, $k = 0, \ldots, N-1$):

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cdot \mathrm{e}^{\mathrm{j}\frac{2\pi kn}{N}} \qquad n = 0, 1, 2, \ldots, N-1$$

Spacing of DT frequencies (grid):

$$\Delta\Omega = \frac{2\pi}{N}$$

DFT values ($X(\Omega)$: DTFT):

$$X[k] = X\left(k \cdot \frac{2\pi}{N}\right)$$

Errors in a computed DFT:

- Aliasing error: control through choice of F_s
- Smearing error: control through choice of N, window
- Grid error: control through choice of N, zero padding

DT Fourier System Models

Frequency Response

Steady state response to sinusoidal inputs ($H(\Omega)$: DTFT):

$$x[n] = A \cdot \cos(\Omega_0 n + \theta)$$
$$\rightarrow\; y[n] = |H(\Omega_0)| A \cdot \cos\left(\Omega_0 n + \theta + \angle H(\Omega_0)\right)$$

Impulse response of an M point moving average filter:

$$h[n] = \begin{cases} \frac{1}{M}, & n = 0, 1, \ldots, M-1 \\ 0, \text{otherwise} & \end{cases}$$

Difference equation for M point MA filter:

$$y[n] = \frac{1}{M} \cdot (x[n] + x[n-1] + \cdots + x[n-M+1])$$

Z Transformation Models for DT Signals & Systems

Transfer Function

Definition of the one sided z-transform for causal signals ($x[n] = 0,\ n < 0$):

$$X(z) = \sum_{n=0}^{\infty} x[n] \cdot z^{-n}$$

Inverse z-transform:

- Partial fractions
- Tables

Convolution property:

$$x[n] * h[n] \circ\!\!-\!\!\bullet\, X(z) \cdot H(z)$$

Right shift for causal signal ($x[n] = 0,\ n < 0,\ x[n] \circ\!\!-\!\!\bullet\, X(z)$):

$$x[n-q] \circ\!\!-\!\!\bullet\, z^{-q} \cdot X(z)$$

Transfer function:

$$H(z) = \frac{Y(z)}{X(z)}$$

Relationship between z-transform and DTFT:

$$X(\Omega) = X(z)|_{z=\mathrm{e}^{\mathrm{j}\Omega}}$$

[1] https://github.com/m-thu/sandbox/blob/master/dsp.tex

Circular convolution:

$$IDFT\{X[k]\cdot H[k]\} = x[n] \circledast h[n]$$

Convert cyclic convolution to linear convolution (L: length of $x[n]$, M: length of $h[n]$):

Zero-pad both signals to length $N \geq L+M-1$,
choose $N = 2^r$ for minimum r

Windowing ($x[n]$: input signal, $w[n]$: window function, $x_N[n]$: truncated signal):

$$X_N[n] = x[n]\cdot w[n]$$

DTFT of windowed signal ($W(\Omega)$: kernel function):

$$X_N(\Omega) = X(\Omega) * W(\Omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\lambda)\cdot W(\Omega-\lambda)\,\mathrm{d}\lambda$$

Rectangular window (length N):

$$w_{\mathrm{R}}[n] = \begin{cases} 1, & 0 \leq n \leq N-1, \\ 0, & \text{otherwise} \end{cases}$$

Kernel function for rectangular window (Dirichlet kernel):

$$W_{\mathrm{R}}(\Omega) = \mathrm{e}^{-\mathrm{j}\Omega\,\frac{N-1}{2}}\cdot\frac{\sin\left(\Omega\,\frac{N}{2}\right)}{\sin\left(\frac{\Omega}{2}\right)}$$

Distortions due to windowing:

- Side-lobe interferences
- Smearing due to width of the main lobe

Properties of window functions (main lobe width, largest side lobe):

- Rectangular window:
  - $\frac{4\pi}{N}$
  - $\approx -13\,\mathrm{dB}$
  - Undesirable side lobes
- Bartlett (Triangular) window:
  - $\frac{8\pi}{N}$
  - $\approx -27\,\mathrm{dB}$
- Hann (Hanning, Cosine) window:
  - $\frac{8\pi}{N}$
  - $-32\,\mathrm{dB}$
- Hamming (Raised Cosine) window:
  - $\frac{8\pi}{N}$
  - $-43\,\mathrm{dB}$
  - Flat kernel function
- Blackman windows:
  - $\frac{12\pi}{N}$
  - $-57\,\mathrm{dB}$
- Kaiser window:
  - $\alpha = 3$: $\frac{6\pi}{N}$, $-25\,\mathrm{dB}$
  - $\alpha = 6$: $\frac{8\pi}{N}$, $-45\,\mathrm{dB}$
  - $\alpha = 9$: $\frac{12\pi}{N}$, $-65\,\mathrm{dB}$
  - $\alpha = 12$: $\frac{16\pi}{N}$, $-90\,\mathrm{dB}$

Measuring amplitude ($w[n]$: window function):

$W(0) = \sum_{n=0}^{N-1} w[n]$
$\rightarrow$ Amplitude: $A\cdot W(0)$

Non-recursive filters (FIR):

$y[n] = \sum_{k=0}^{M} b_k\cdot x[n-k]$
$H(z) = b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}$
$\rightarrow H(z) = \frac{b_0 z^M + b_1 z^{M-1} + \cdots + b_M}{z^M}$

Impulse invariance ($h(t)$: impulse response of a CT system, T_{s}: sampling interval):

- Equivalent DT system: $h[n] = T_{\mathrm{s}}\cdot h(t = n\cdot T)$
- Not suitable for high-pass filters!
- Implementation as FIR filter.

Tustin's method / Bilinear transformation
($H(s)$: CT transfer function):

- $s \rightarrow \frac{2}{T_{\mathrm{s}}}\cdot\frac{z-1}{z+1} = \frac{2}{T_{\mathrm{s}}}\cdot\frac{1-z^{-1}}{1+z^{-1}}$
- Implementation as IIR filter.
- Frequency distortion in amplitude and phase response, but no aliasing.
- $\Omega = 2\,\arctan\left(\frac{T_{\mathrm{s}}}{2}\,\omega\right)$, $\omega = \frac{2}{T_{\mathrm{s}}}\,\tan\left(\frac{\Omega}{2}\right)$
  $-\infty < \omega < \infty$ is mapped into $-\pi < \Omega < \pi$.

Misc

Geometric sum:

$$\sum_{n=q_1}^{q_2} = \frac{r^{q_1} - r^{q_2+1}}{1-r}$$

DTFT transform ($|a| < 1$):

$$a^n\,u[n] \circ\!\!-\!\!\bullet \frac{1}{1-a\cdot\mathrm{e}^{-\mathrm{j}\Omega}}$$

DTFT multiplication by cosine:

$$\cos(\Omega_0 n)\cdot x[n] \circ\!\!-\!\!\bullet \frac{1}{2}\left[X(\Omega+\Omega_0) + X(\Omega-\Omega_0)\right]$$

DTFT multiplication by sine:

$$\sin(\Omega_0 n)\cdot x[n] \circ\!\!-\!\!\bullet \frac{\mathrm{j}}{2}\left[X(\Omega+\Omega_0) - X(\Omega-\Omega_0)\right]$$

Z transform ($a \in \mathbb{C}$):

$$a^n\,u[n] \circ\!\!-\!\!\bullet \frac{z}{z-a}$$